Int. J. Solids Structures, 1977, Vol. 13, pp. 1081–1089. Pergamon Press. Printed in Great Britain

NONLINEAR ANALYSIS OF MULTILAYERED SHELLS†

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(Received 6 December 1976; revised 13 April 1977)

Abstract—A large deformation theory for layered shells of arbitrarily varying thickness and using a piecewise smooth displacement field is developed. A system of layer coordinates is introduced which allows the results to be presented in a simple compact form analogous to the theory of monocoque shells.

NOTATION

a determinant of the reference surface metric
$\bar{a}_\alpha$ natural base vectors on the reference surface
$a_{\alpha\beta}$ metric tensor of the reference surface
A determinant of the three-dimensional metric
$\bar{A}_i$ natural base vectors
A_{ij} three dimensional metric tensor
$B^{\alpha\beta JK}$ stress integral, eqn (10c)
C boundary curve of the reference surface
$\bar{d}_I$ director of the Ith layer
$D_I^{ijkIJ\ldots L}$ see eqn (A2)
$e_{ij}, e_{ijJ}, e_{ijJK}$ see eqns (A4) and (6)
E_I^{ijkl} elastic moduli of the Ith layer, eqn (A1)
EVW external virtual work
h_I length of the Ith director
i, j, k, l tensorial indices with range 1, 2, 3
I, J, K, L, M indices with range $1, \ldots, N$
IVW internal virtual work
$M^{\alpha\beta J}$ stress integral, eqn (10b)
$\bar{n}_I$ unit normal to the boundary surface of the Ith layer
$\bar{N}$ unit normal to the undeformed z-surface
N number of layers
$N^{\alpha\beta}$ stress integral, eqn (10a)
$O_I(P)$ origin of layer I corresponding to P
${}_0\bar{p}$ dead load per unit area of the outer surface of the undeformed shell
P point on the reference surface, Fig. 2
$\bar{P}^I$ stress integral, eqn (10f)
$\bar{q}$ dead load per unit area of the boundary surface of the undeformed shell
Q point off the reference surface, Fig. 2
$Q^{\alpha IJ}$ stress integral, eqn (10e)
$\bar{r}$ position vector of P
$\bar{R}$ position vector of Q
S area of the undeformed reference surface
S_I boundary surface of the ith layer
$S^{\alpha I}$ stress integral, eqn (10d)
$\bar{u}$ displacement vector of P
V volume of the shell
V_I volume of the Ith layer
x^α reference surface coordinates
z^I layer coordinates
$\alpha, \beta, \lambda, \mu$ tensorial indices with range 1, 2
ϵ_{ij} strain tensor components
σ^{ij} 2nd Piola–Kirchhoff stress tensor components.

The asterisk (*) is used to denote quantities referring to the deformed configuration.

1. INTRODUCTION

It is well recognized that sandwich and laminated elements are more complex than the traditional single material elements. Therefore, more sophisticated models have to be used in predicting their behaviour. For example, in soft-core sandwich construction it is imperative that

†This work was supported in part by the National Research Council of Canada, through a research grant to the second author, No. A-2736.

the transverse effects be included and account be taken of the discontinuities at the interfaces between layers. Considerable research work, dealing with the behaviour of composites, has been carried out in the recent past to refine analytical methods and allow more efficient and economical designs.

A survey of work related to composite material mechanics has been published recently in [1]. An approximate theory for plane laminated media composed of alternating matrix and fiber-reinforcing layers has been presented[2] and extended to the case of curvilinear laminated composites[3]. In such a treatment the displacements in each layer are first expressed as two-term expansions about the respective mid-planes and then, by means of a smoothing operation, the layered medium is replaced by a homogeneous continuum. It has been shown[4, 5] that good agreement with predictions from an exact linear three-dimensional analysis of laminated plates can be obtained if the displacement components are assumed to be piecewise linear across the thickness. In [6] such an assumption was adopted, but the normal displacement was assumed to be constant across the thickness, thus neglecting transverse normal strain effects. The results obtained from this kind of analysis lead some authors to believe that similar assumptions, namely a continuous displacement field across the thickness which is not necessarily smooth at the interfaces of the various layers, may yield sufficiently reliable results for the non-linear analysis of multi-layered shells.

This paper presents a non-linear theory on the statics of multilayered shells, including transverse effects. The approach adopted[7, 8] is purely kinematical. The displacement field is assumed to belong to a certain finite-parameter family of functions while the "exact" three-dimensional kinematic relations and constitutive equations are used. Stresses and strains, rather than stress resultants and associated kinematic variables, are used in formulating the principle of virtual work, from which the field equations and relevant boundary conditions are obtained. Stress resultants appear only formally in the equations and their number and nature is a direct consequence of the choice of the functions for the displacement field. In particular, if these functions are polynomials in the thickness coordinate, the stress integrals turn out to be moments of various orders. The dynamic boundary conditions are also determined by the kinematic hypothesis used. For a prescribed distribution of boundary tractions, the kinematic assumption acts as a filter that admits certain integral properties of the tractions and ignores all others. For example, the classical Kirchhoff–Love hypothesis results in the recognition of only some zero-th and first order moments.

The simplest kinematic hypothesis which still accounts for transverse effects is a piecewise linear displacement field. For this reason, and for the sake of brevity, the equations resulting from such a first order theory are given here in detail. This kind of treatment of multilayered shells is, in some respects, analogous to a multidirector Cosserat surface theory, to be treated elsewhere.

Vector and tensor notations are used throughout. In addition, a novel non-tensorial index notation is employed which facilitates the treatment of all layers in a compact manner. All Greek indices range from 1 to 2, all lower case Latin indices range from 1 to 3 and all upper case Latin indices range from 1 through N.

2. SHELL GEOMETRY

Directors

Let the shell space be composed of N distinct layers of variable thickness. It is convenient to think of each layer as a two-dimensional set of directed straight material line segments both ends of which describe smooth surfaces in E_3. As in some idealized sandwich shell theory treatments[9, 10] these directed material segments may be called directors $\bar{d}_I$ ($I = 1, \ldots, N$), a term borrowed from Cosserat surface theory[11, 12]. All directors are assumed to have the same general orientation, with no "tip to tip" or "tail to tail" contact between directors of adjacent layers.

Reference surface

The reference surface is assumed to be a smooth surface contained completely within a single layer. To each point P of the reference surface there corresponds a material line, generally zig-zag, composed of the N directors continuously connected to P. This material line will be called the directrix at P. A multilayered shell may thus be regarded as a two-dimensional collection of directrices.

Let x^α ($\alpha = 1, 2$) be a parametrization of the reference surface. This parametrization refers also naturally to the directrix field and therefore to the director fields of all layers.

Layer coordinates

A set of non-dimensional layer coordinates, z^I ($I = 1, \ldots, N$), is defined for each point of the directrix at P in the following manner:

(a) Starting at P and following the directrix, the first point of layer I encountered is called the origin, $O_I(P)$, of that layer corresponding to P. Thus the local origin of any layer is either the "tail" or the "tip" of its director which is part of the directrix at P, except for the layer containing the reference surface, for which the origin is P itself.

(b) At layer I, the value of z^J is evaluated as follows: (i) for $J = I$: $z^J|\bar{d}_I|$ measures length along $\bar{d}_I$, starting from zero at the origin $O_I(P)$ and increasing in the positive sense of $\bar{d}_I$; (ii) for $J \neq I$: z^J is a constant equal to the last value attained by z^J in layer J when travelling from P towards layer I, or zero if layer J is not traversed in that trajectory.

This definition of layer coordinates is illustrated in Fig. 1 for a shell composed of 5 layers. Note that each of the five coordinate plots depicts the variation of a coordinate through all layers. Thus the five coordinates of any point along the directrix are the values indicated by the intersection of a horizontal line through that point with the coordinate plots. Any such horizontal line will intersect only one coordinate plot in its linearly varying range, while the other four plots will each be cut at a range of constant value of its coordinate. The only point for which all five coordinates vanish is the point P on the reference surface.

Upon reflection it becomes clear that such a layer coordinate definition provides a convenient method for accumulating the thicknesses of the inner layers in the expression of the position vector for a point off the reference surface. A similar coordinate definition was used in [13] for the nonlinear treatment of beams with open thin-walled cross-section.

Metric properties

The position of a point Q in the undeformed shell space is defined by the position vector $\bar{R}$ as:

$$\bar{R} = \bar{r}(x^\alpha) + z^I \bar{d}_I(x^\alpha) \tag{1}$$

where $\bar{r}$ denotes the position vector of point P (Fig. 2) on the reference surface. It is important to note that although the summation convention applies to diagonally repeated upper case latin indices, these indices are not tensorial. This slight deviation is the penalty for the convenience of the method of derivation and the compactness of the results achieved from the introduction of the layer coordinates definition. Note the formal agreement of eqn (1) with corresponding expressions for monocoque shells. This similarity carries through all details of the sequel where upper case latin indices, with range 1 through N, take the place of the single index 3.

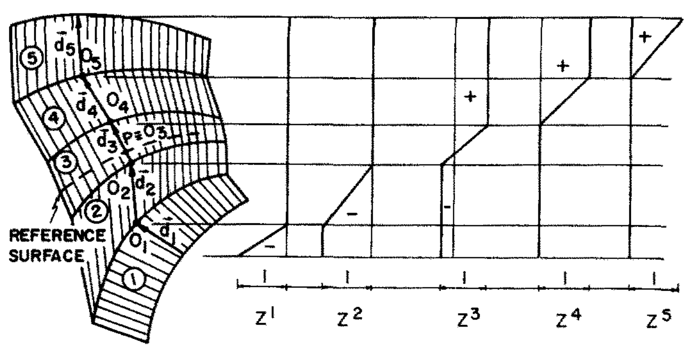

Fig. 1. Definition of layer coordinates for 5-layer shell.

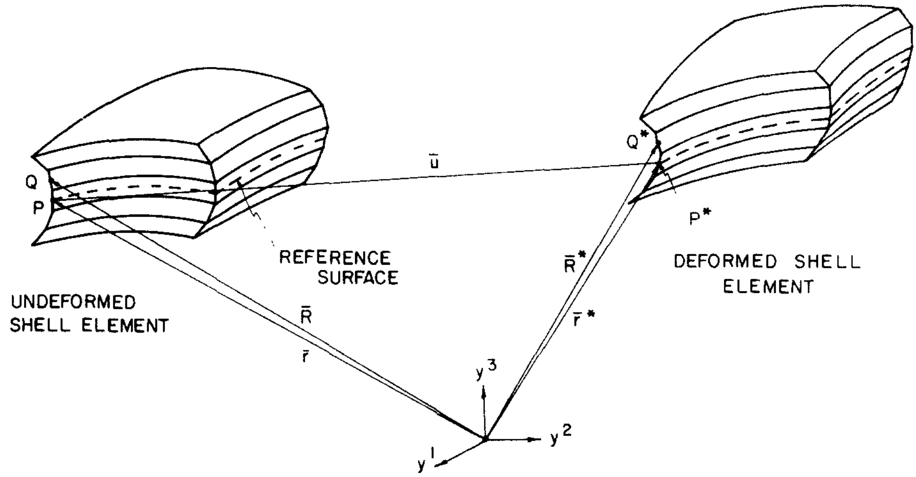

Fig. 2. Multilayered shell space deformation.

At any point of layer I, the natural basis is obtained by differentiating (1) with respect to the surface coordinates, x^α, and the layer coordinate z^I, as follows:

$$\bar{A}_\alpha = \bar{R}_{,\alpha} = \bar{r}_{,\alpha} + z^J \bar{d}_{J,\alpha} = \bar{a}_\alpha + z^J \bar{d}_{J,\alpha} \quad (2a)$$

$$(\bar{A}_3)_I = \frac{\partial \bar{R}}{\partial z^I} = \bar{d}_I. \quad (2b)$$

Clearly, at the interface between two layers there is a non-uniqueness in the third base vector. Note that for the special case of all directors coinciding with the normal to the reference surface and all layers being of constant thickness, eqn (2a) reduces to the corresponding expression of classical thin shell theory. Also, for the case of a single layer this treatment reduces to that of a non-normal coordinate treatment of thin shell theory [14, 15].

The components of the metric tensor for a point in layer I are obtained by taking the dot products between the base vectors as follows:

$$A_{\alpha\beta} = \bar{a}_\alpha \cdot \bar{a}_\beta + z^J(\bar{a}_\alpha \cdot \bar{d}_{J,\beta} + \bar{a}_\beta \cdot \bar{d}_{J,\alpha}) + z^J z^K \bar{d}_{J,\dot{\alpha}} \cdot \bar{d}_{K,\beta} \quad (3a)$$

$$(A_{\alpha 3})_I = (A_{3\alpha})_I = \bar{a}_\alpha \cdot \bar{d}_I + z^J \bar{d}_{J,\alpha} \cdot \bar{d}_I \quad (3b)$$

$$(A_{33})_I = \bar{d}_I \cdot \bar{d}_I = h_I^2 \quad (3c)$$

where $\bar{a}_\alpha \cdot \bar{a}_\beta = a_{\alpha\beta}$ is the metric tensor of the reference surface and where h_I is the length of $\bar{d}_I$.

For the particular case of a straight directrix normal to the reference surface and for layers of constant thickness, eqns (3) reduce to the familiar expressions of thin shell theory.

3. KINEMATICS

Kinematic hypothesis

As stated above, the approach used here is a purely kinematical one in the sense that it is based on an assumption regarding the dependence of the displacement field on the thickness parameter. The "exact" three-dimensional kinematic relations and constitutive equations are used in conjunction with the principle of virtual work, eliminating the need for any further approximations or assumptions in the theory.

In the present paper the simplest possible kinematic hypothesis admitting transverse shear and normal strains is adopted. This hypothesis assumes the displacement field to depend linearly on the layer coordinates z^J ($J = 1, \ldots, N$), or, equivalently, the position vector, $\bar{R}^*$, of a point Q^* in the deformed shell to be given by

$$\bar{R}^* = \bar{r} + \bar{u} + z^J \bar{d}_J^* \quad (4)$$

where $\bar{u}$ denotes the displacement of point P of the reference surface (Fig. 2) and $\bar{d}_J^*$ is the deformed counterpart of $\bar{d}_J$, assumed to remain straight after deformation. This is equivalent to assuming piecewise linear variation of all displacement components across the thickness.

Metric properties of the deformed shell space

Considering x^α and z^J as convected coordinates yields the following vectors as the deformed counterparts to $\bar{A}_\alpha$ and $\bar{A}_3$ at layer I:

$$\bar{A}_\alpha^* = \bar{a}_\alpha + \bar{u}_{,\alpha} + z^J \bar{d}_{J,\alpha}^* \tag{5a}$$

$$(\bar{A}_3^*)_I = \bar{d}_I^*. \tag{5b}$$

Following the procedure used in Section 2, expressions for $A_{\alpha\beta}^*$, $A_{\alpha 3}^*$ and A_{33}^*, analogous to those given in eqns (3), are derived, from which the components of the strain tensor for layer I are obtained as

$$\begin{aligned}\epsilon_{\alpha\beta} &= \frac{1}{2}(A_{\alpha\beta}^* - A_{\alpha\beta}) = \frac{1}{2}[\bar{a}_\alpha \cdot \bar{u}_{,\beta} + \bar{a}_\beta \cdot \bar{u}_{,\alpha} + \bar{u}_{,\alpha} \cdot \bar{u}_{,\beta}] \\ &+ \frac{1}{2} z^J [\bar{a}_\alpha \cdot (\bar{d}_J^* - \bar{d}_J)_{,\beta} + \bar{a}_\beta \cdot (\bar{d}_J^* - \bar{d}_J)_{,\alpha} + \bar{u}_{,\alpha} \cdot \bar{d}_{J,\beta}^* + \bar{u}_{,\beta} \cdot \bar{d}_{J,\alpha}^*] \\ &+ \frac{1}{2} z^J z^K [\bar{d}_{J,\alpha}^* \bar{d}_{K,\beta}^* - \bar{d}_{J,\alpha} \cdot \bar{d}_{K,\beta}]\end{aligned} \tag{6a}$$

$$\begin{aligned}(\epsilon_{\alpha 3})_I &= \frac{1}{2}[(A_{\alpha 3}^*)_I - (A_{\alpha 3})_I] \\ &= \frac{1}{2}[\bar{a}_\alpha \cdot (\bar{d}_I^* - \bar{d}_I) + \bar{u}_{,\alpha} \cdot \bar{d}_I^*] + \frac{1}{2} z^J (\bar{d}_{J,\alpha}^* \cdot \bar{d}_I^* - \bar{d}_{J,\alpha} \bar{d}_I)\end{aligned} \tag{6b}$$

$$(\epsilon_{33})_I = \frac{1}{2}[(A_{33}^*)_I - (A_{33})_I] = \frac{1}{2}[\bar{d}_I^* \cdot \bar{d}_I^* - \bar{d}_I \cdot \bar{d}_I] \tag{6c}$$

Note that for zero normal strain in layer I, i.e. $\epsilon_{33} \equiv 0$, the shearing strain components, $\epsilon_{\alpha 3}$, are easily verified to be constant along the director of that layer.

4. FIELD EQUATIONS

Virtual work

The internal virtual work in a three-dimensional body is expressed as

$$IVW = \int_V \sigma^{ij} \delta\epsilon_{ij} \, \mathrm{d}V \quad (i, j = 1, 2, 3) \tag{7}$$

where σ^{ij} is the symmetric or second Piola–Kirchhoff stress tensor, ϵ_{ij} is the Lagrangean strain tensor and V designates the volume of the undeformed body. For a multilayered shell, this expression takes the form

$$IVW = \sum_{I=1}^{N} \int_{V_I} (\sigma^{\alpha\beta} \delta\epsilon_{\alpha\beta} + 2\sigma^{\alpha 3} \delta\epsilon_{\alpha 3} + \sigma^{33} \delta\epsilon_{33}) \, \mathrm{d}V_I \tag{8}$$

where V_I is the volume of the Ith layer in the reference configuration.

Introducing eqns (6) into (8), one obtains

$$\begin{aligned}IVW = \int_S \{ & N^{\alpha\beta} (\bar{a}_\alpha + \bar{u}_{,\alpha}) \cdot \delta\bar{u}_{,\beta} + M^{\alpha\beta J} [(\bar{a}_\alpha + \bar{u}_{,\alpha}) \cdot \delta\bar{d}_{J,\beta}^* + \bar{d}_{J,\beta}^* \cdot \delta\bar{u}_{,\alpha}] + B^{\alpha\beta JK} \bar{d}_{J,\alpha}^* \cdot \delta\bar{d}_{K,\beta}^* \\ & + S^{\alpha I} [(\bar{a}_\alpha + \bar{u}_{,\alpha}) \cdot \delta\bar{d}_I^* + \bar{d}_I^* \cdot \delta\bar{u}_{,\alpha}] + Q^{\alpha IJ} [\bar{d}_I^* \cdot \delta\bar{d}_{J,\alpha}^* + \bar{d}_{J,\alpha}^* \cdot \delta\bar{d}_I^*] + \bar{P}^I \cdot \delta\bar{d}_I^* \} \, \mathrm{d}S\end{aligned} \tag{9}$$

where S is the area of the undeformed reference surface and where the following notation is used:

$$N^{\alpha\beta} = \sum_{I=1}^{N} \int \sqrt{\frac{A}{a}} \sigma^{\alpha\beta} \, dz^I \tag{10a}$$

$$M^{\alpha\beta J} = \sum_{I=1}^{N} \int \sqrt{\frac{A}{a}} \sigma^{\alpha\beta} z^J \, dz^I \tag{10b}$$

$$B^{\alpha\beta JK} = \sum_{I=1}^{N} \int \sqrt{\frac{A}{a}} \sigma^{\alpha\beta} z^J z^K \, dz^I \tag{10c}$$

$$S^{\alpha I} = \int \sqrt{\frac{A}{a}} \sigma^{\alpha 3} \, dz^I \tag{10d}$$

$$Q^{\alpha IJ} = \int \sqrt{\frac{A}{a}} \sigma^{\alpha 3} z^J \, dz^I \tag{10e}$$

$$\bar{P}^I = \bar{d}^I_{\mathfrak{z}} \int \sqrt{\frac{A}{a}} \sigma^{33} \, dz^I \tag{10f}$$

with

$$A = \det [A_{ij}] \tag{11a}$$

$$a = \det [a_{\alpha\beta}]. \tag{11b}$$

Equations (10) represent stress resultants of various kinds. Any given elastic constitutive equation, i.e. an expression of the form

$$\sigma^{ij} = F^{ij}(\epsilon_{kl}) \tag{12}$$

can be transformed, by means of eqns (6), into a functional relationship between the stress and the kinematic variables as

$$\sigma^{ij} = F^{ij}[\epsilon_{kl}(\bar{u}, \bar{d}^I_{\mathfrak{z}})] = \Phi^{ij}[\bar{u}, \bar{d}^I_{\mathfrak{z}}]. \tag{13}$$

Upon substitution of (13) into eqns (10), the various stress resultants become functions of the kinematic variables only. As an example, these functions are exhibited in Appendix A for the most general case of linearly elastic layers.

Consider now the virtual work of the external forces. For the sake of simplicity, only two types of loading are included: (a) dead load, ${}_o\bar{p}$, acting on the outer surface of the shell, and (b) dead load, $\bar{q}$, acting on the boundary surface of the shell (Fig. 3).

The virtual work of the first type of loading is given by

$$EVW_1 = \int_{{}_oS} {}_o\bar{p} \cdot (\delta\bar{u} + {}_oz^J \delta\bar{d}^J_{\mathfrak{z}}) \, d_oS \tag{14}$$

where the left subscript "o" refers to the outer surface. This expression is easily transformed into an integral over the reference surface, yielding

$$EVW_1 = \int_S \sqrt{\frac{{}_oa}{a}} {}_o\bar{p} \cdot (\delta\bar{u} + {}_oz^J \delta\bar{d}^J_{\mathfrak{z}}) \, dS \tag{15}$$

in which ${}_oa$ denotes the determinant of $A_{\alpha\beta}$ evaluated at the outer surface.

The virtual work of the forces acting on the boundary surface is expressed as

$$EVW_2 = \sum_{I=1}^{N} \int_{S_I} \bar{q} \cdot (\delta\bar{u} + z^J \delta\bar{d}^J_{\mathfrak{z}}) \, dS_I \tag{16}$$

where S_I is the boundary surface of the Ith layer. Denoting by $\bar{n}_I$ the unit normal to S_I, and by

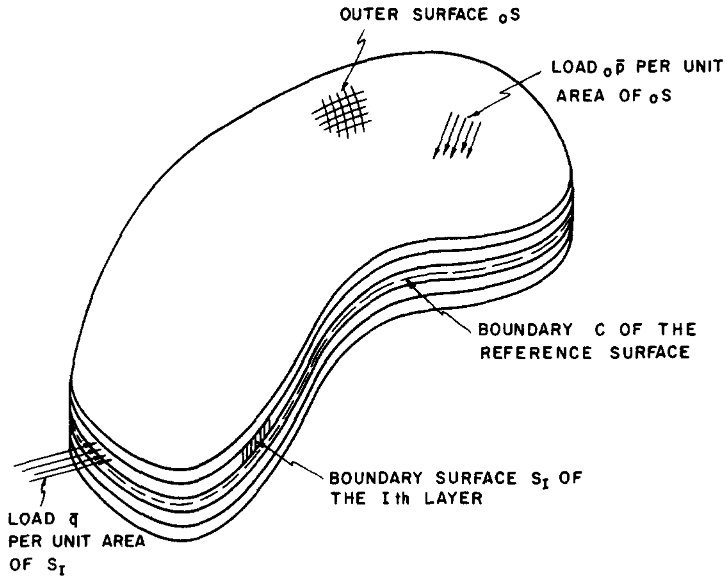

Fig. 3. Shell loading.

C the boundary of the reference surface, eqn (16) may be written as

$$EVW_2 = \oint_C \sum_{I=1}^{N} \int_{z^I} \bar{q} \cdot (\delta\bar{u} + z^J \delta\bar{d}^*_J)[\bar{n}_I \cdot (\bar{A}_\alpha \times \bar{d}_I)] \frac{\mathrm{d}x^\alpha}{\mathrm{d}C} \mathrm{d}z^I \, \mathrm{d}C. \tag{17}$$

Note that from eqn (2a) it follows that the mixed product $\bar{n}_I \cdot (\bar{A}_\alpha \times \bar{d}_I)$ is linear in z^I.

Equilibrium equations

The principle of virtual work is stated as

$$IVW = EVW = EVW_1 + EVW_2. \tag{18}$$

Since no constraint has been imposed, all variations appearing in the virtual work expressions are mutually independent. Therefore, from eqn (18), the equilibrium equations are obtained as

$$-[N^{\beta\alpha}(\bar{a}_\beta + \bar{u}_{,\beta}) + M^{\alpha\beta J}\bar{d}^*_{J,\beta} + S^{\alpha J}\bar{d}^*_J]|_\alpha = \sqrt{\frac{_0a}{a}}\,{}_0\bar{p} \tag{19a}$$

$$-[M^{\alpha\beta I}(\bar{a}_\alpha + \bar{u}_{,\alpha}) + B^{\alpha\beta JI}\bar{d}^*_{J,\alpha} + Q^{\beta JI}\bar{d}^*_J]|_\beta + S^{\alpha I}(\bar{a}_\alpha + \bar{u}_{,\alpha}) + Q^{\alpha IJ}\bar{d}^*_{J,\alpha} + \bar{P}^I = \sqrt{\frac{_0a}{a}}\,{}_0z^I{}_0\bar{p} \tag{19b}$$

where "|" denotes covariant differentiation with respect to the undeformed metric. Note that eqn (19a) represents 3 component equations, whereas eqn (19b) represents $3N$ component equations.

Boundary conditions

The boundary conditions associated with eqns (19) are

$$\left\{\begin{aligned} &[N^{\beta\alpha}(\bar{a}_\beta + \bar{u}_{,\beta}) + M^{\alpha\beta J}\bar{d}^*_{J,\beta} + S^{\alpha J}\bar{d}^*_J]\nu_\alpha \\ &\qquad = \sum_{I=1}^{N}\int_{z^I} \bar{q}[\bar{n}_I \cdot (\bar{A}_\alpha \times \bar{d}_I)]\frac{\mathrm{d}x^\alpha}{\mathrm{d}C}\mathrm{d}z^I \\ &\text{or} \\ &\bar{u} \text{ prescribed} \end{aligned}\right. \tag{20a}$$

$$\left\{\begin{aligned}&[M^{\alpha\beta I}(\bar{a}_\alpha+\bar{u}_{,\alpha})+B^{\alpha\beta JI}\bar{d}^*_{J,\alpha}+Q^{\beta JI}\bar{d}^*_J]\nu_\beta\\&\quad=\sum_{J=1}^{N}\int_{z^J}\bar{q}z^I[\bar{n}_J\cdot(\bar{A}_\alpha\times\bar{d}_J)]\frac{\mathrm{d}x^\alpha}{\mathrm{d}C}\,\mathrm{d}z^J\\&\text{or}\\&\bar{d}^*_I\text{ prescribed}\end{aligned}\right. \tag{20b}$$

where ν_α is the unit normal to the boundary curve on the reference surface. The "or" alternatives mean that if a component of, say, $\bar{u}$ is prescribed, then the corresponding component of the force boundary condition is dropped. Alternatively, appropriate linear combinations of forces and displacements may be prescribed.

CONCLUSIONS

A geometrically non-linear theory for multi-layered shells, made of elastic materials, has been presented. More general constitutive laws could be accommodated.

The entire treatment is relatively compact and simple, and analogous to the single-layer treatment of classical shell theory. This compactness and simplicity is made possible by the introduction of a system of layer coordinates.

Although no numerical results are presented, the results obtained lend themselves to the formulation of multi-layer sandwich shell theory, in which the soft "idealized" cores are separated by membrane layers. Furthermore, the present treatment of multi-layered shells suggests some analogy with a multi-polar Cosserat surface model of layered shells. Such an analogy, and the differences between the theory derived here and a multi-polar Cosserat surface treatment will be the subject of a further study[16].

Acknowledgements—The financial assistance provided by the National Research Council of Canada for this work, in the form of a research grant to the second author, Grant No. A-2736, is gratefully acknowledged.

REFERENCES

1. C. W. Bert and P. H. Francis, Composite material mechanics: structural mechanics. *AIAA J.* **12**, 1173–1186 (1974).
2. C. T. Sun, J. D. Achenbach and G. Herrmann, Continuum theory for a laminated medium. *J. Appl. Mech.* **35**, 467–475 (1968).
3. R. A. Grot, A continuum model for curvilinear laminated composites. *Int. J. Solids Structures* **8**, 439–462 (1972).
4. N. J. Pagano, Exact solutions for rectangular bidirectional composites and sandwich plates. *J. Composite Materials* **4**, 20–36 (1970).
5. S. Srinivas and A. K. Rao, Flexure of thick rectangular plates. *J. Appl. Mech.* **40**, 298–299 (1973).
6. S. Srinivas, A refined analysis of composite laminates. *J. Sound and Vibration* **30**, 495–507 (1973).
7. M. Epstein, General kinematic approach to the nonlinear analysis of structures, with applications to shell theory, (in Hebrew). Thesis, Israel Institute of Technology, Haifa (1972).
8. Y. Tene and M. Epstein, A direct variational formulation of nonlinear shell theory. *Proc. of the Int. Conf. on Variational Methods in Engng*, pp. 7/47–7/54. Southampton University, England (1972).
9. P. G. Glockner and D. J. Malcolm, A Lagrangian formulation of sandwhich shell theory. *ASCE J. Engng Mech. Div.* **99** (No. EM3, Proc. Paper 9775), 445–466 (June 1973).
10. D. J. Malcolm and P. G. Glockner, Variational principles for idealized sandwich shells. *Int. J. Engng. Sci.* **14**, 661–671 (1976).
11. H. Cohen and C. N. DeSilva, Nonlinear theory of elastic directed surfaces. *J. Math. Phys.* **7**, 960–966 (1966).
12. A. E. Green, P. M. Naghdi and W. L. Wainwright, A general theory of a Cosserat surface. *Arch. Rat. Mech. Anal.* **20**, 287–308 (1965).
13. M. Epstein and D. W. Murray, Large three-dimensional deformations of thin-walled beams. *Int. J. Solids Structures* **12**, 867–879 (1976).
14. M. Epstein and Y. Tene, A linear theory of thin elastic shells, based on conservation of a non-normal straight line. *Int. J. Solids Structures* **9**, 257–268 (1973).
15. D. J. Malcolm and P. G. Glockner, Nonlinear sandwich shell and Cosserat surface theory. *ASCE J. Engng Mech. Div.* **98** (No. EM5, Proc. Paper 9244), 1183–1203 (1972).
16. M. Epstein and P. G. Glockner, Multilayered shells and directed surfaces. Departmental Report, Dept. of Mech. Engng, University of Calgary (1977).

APPENDIX A

Explicit form of the stress integrals for the case of linearly elastic layers

Let the constitutive equation for the Ith layer be given by

$$\sigma^{ij}=E_I^{ijkl}\epsilon_{kl} \tag{A1}$$

where the elastic moduli E_I^{ijkl} depend on x^α and z^I.

Introduce the following notation:

$$D_I^{ijklJ\ldots L} = \int_{z^I} \sqrt{\frac{A}{a}} E_I^{ijkl} \quad z^J \ldots z^L \, dz^I. \tag{A2}$$

These integrals are evaluated over the undeformed shell.

Note that the strain components, eqns (6), have the form

$$\epsilon_{\alpha\beta} = e_{\alpha\beta} + z^J e_{\alpha\beta J} + z^J z^K e_{\alpha\beta JK} \tag{A3a}$$

$$(\epsilon_{\alpha 3})_I = (e_{\alpha 3})_I + z^J (e_{\alpha 3J})_I \tag{A3b}$$

$$(\epsilon_{33})_I = (e_{33})_I \tag{A3c}$$

where the e's are surface tensors, as indicated by their Greek indices, defined by

$$e_{\alpha\beta} = \frac{1}{2}[\bar{a}_\alpha \cdot \bar{u}_{,\beta} + \bar{a}_\beta \cdot \bar{u}_{,\alpha} + \bar{u}_{,\alpha} \cdot \bar{u}_{,\beta}] \tag{A4a}$$

$$e_{\alpha\beta J} = \frac{1}{2}[\bar{a}_\alpha \cdot (\bar{d}_J^* - \bar{d}_J)_{,\beta} + \bar{a}_\beta \cdot (\bar{d}_J^* - \bar{d}_J)_{,\alpha} + \bar{u}_{,\alpha} \cdot \bar{d}^*_{J,\beta} + \bar{u}_\beta \cdot \bar{d}^*_{J,\alpha}] \tag{A4b}$$

$$e_{\alpha\beta JK} = \frac{1}{2}[\bar{d}^*_{J,\alpha} \cdot \bar{d}^*_{K,\beta} - \bar{d}_{J,\alpha} \cdot \bar{d}_{K,\beta}] \tag{A4c}$$

$$(e_{\alpha 3})_I = \frac{1}{2}[\bar{a}_\alpha \cdot (\bar{d}_I^* - \bar{d}_I) + \bar{u}_{,\alpha} \cdot \bar{d}_I^*] \tag{A4d}$$

$$(e_{\alpha 3J})_I = \frac{1}{2}[\bar{d}^*_{J,\alpha} \cdot \bar{d}_I^* - \bar{d}_{J,\alpha} \cdot \bar{d}_I] \tag{A4e}$$

$$(e_{33})_I = \frac{1}{2}[\bar{d}_I^* \cdot \bar{d}_I^* - \bar{d}_I \cdot \bar{d}_I] \tag{A4f}$$

From eqns (10) it follows that

$$\begin{aligned} N^{\alpha\beta} = \sum_{I=1}^{N} [&D_I^{\alpha\beta\lambda\mu} e_{\lambda\mu} + D_I^{\alpha\beta\lambda\mu J} e_{\lambda\mu J} + D_I^{\alpha\beta\lambda\mu JK} e_{\lambda\mu JK} \\ &+ 2D_I^{\alpha\beta\lambda 3}(e_{\lambda 3})_I + 2D_I^{\alpha\beta\lambda 3J}(e_{\lambda 3J})_I + D_J^{\alpha\beta 33}(e_{33})_I] \end{aligned} \tag{A5a}$$

$$\begin{aligned} M^{\alpha\beta J} = \sum_{I=1}^{N} [&D_I^{\alpha\beta\lambda\mu J} e_{\lambda\mu} + D_I^{\alpha\beta\lambda\mu JK} e_{\lambda\mu K} + D_I^{\alpha\beta\lambda\mu JKL} e_{\lambda\mu KL} \\ &+ 2D_I^{\alpha\beta\lambda 3J}(e_{\lambda 3})_I + 2D_I^{\alpha\beta\lambda 3JK}(e_{\lambda 3K})_I + D_I^{\alpha\beta 33J}(e_{33})_I] \end{aligned} \tag{A5b}$$

$$\begin{aligned} B^{\alpha\beta JK} = \sum_{I=1}^{N} [&D_I^{\alpha\beta\lambda\mu JK} e_{\lambda\mu} + D_I^{\alpha\beta\lambda\mu JKL} e_{\lambda\mu L} + D_I^{\alpha\beta\lambda\mu JKLM} e_{\lambda\mu LM} \\ &+ 2D_I^{\alpha\beta\lambda 3JK}(e_{\lambda 3})_I + 2D_I^{\alpha\beta\lambda 3KL}(e_{\lambda 3L})_I + D_I^{\alpha\beta 33JK}(e_{33})_I] \end{aligned} \tag{A5c}$$

$$\begin{aligned} S^{\alpha I} = &D_I^{\alpha 3\lambda\mu} e_{\lambda\mu} + D_I^{\alpha 3\lambda\mu J} e_{\lambda\mu J} + D_I^{\alpha 3\lambda\mu JK} e_{\lambda\mu JK} \\ &+ 2D_I^{\alpha 3\lambda 3}(e_{\lambda 3})_I + 2D_I^{\alpha 3\lambda 3J}(e_{\lambda 3J}) + D_I^{\alpha 333}(e_{33})_I \end{aligned} \tag{A5d}$$

$$\begin{aligned} Q^{\alpha IJ} = &D_I^{\alpha 3\lambda\mu J} e_{\lambda\mu} + D_I^{\alpha 3\lambda\mu JK} e_{\lambda\mu K} + D_I^{\alpha 3\lambda\mu JKL} e_{\lambda\mu KL} \\ &+ 2D_I^{\alpha 3\lambda 3J}(e_{\lambda 3})_I + 2D_I^{\alpha 3\lambda 3JK}(e_{\lambda 3K})_I + D_I^{\alpha 333J}(e_{33})_I \end{aligned} \tag{A5e}$$

$$\begin{aligned} \bar{P}^I = &\bar{d}_I^{33\lambda\mu} e_{\lambda\mu} + D_I^{33\lambda\mu J} e_{\lambda\mu J} + D_I^{33\lambda\mu JK} e_{\lambda\mu JK} \\ &+ 2D_I^{33\lambda 3}(e_{\lambda 3})_I + 2D_I^{33\lambda 3J}(e_{\lambda 3J})_I + D_I^{3333}(e_{33})_I\}. \end{aligned} \tag{A5f}$$